

Locally 2-fold symmetric manifolds are locally symmetric

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Abstract

A manifold is locally k -fold symmetric, if for any point and any k -dimensional vector subspace tangent to this point there exists a local isometry such that this point is a fixed point and the differential of the isometry restricted to that k -dimensional vector subspace is minus the identity. We show that for $k \geq 2$, Riemannian, pseudoriemannian and Finslerian locally k -fold symmetric manifolds are locally symmetric.

Introduction

A Riemannian or Finslerian manifold (M, g) or (M, F) will be called locally k -fold symmetric, if for any point x and for any k -dimensional vector subspace $V^k \subseteq T_x M$ there exists $r > 0$ and a local isometry $\phi : B_r(x) \rightarrow B_r(x)$ of the r -ball around x such that $\phi(x) = x$ and its differential at the point x restricted to V^k is minus the identity. The definition survives in the pseudoriemannian setting (when the metric g is nondegenerate but may have arbitrary signature) if we merely allow the isometry to be defined in some neighborhood of x and not necessary in the r -ball.

If $k = \dim(M)$, then the definition above is one of the standard definitions of locally-symmetric spaces; locally 1-fold symmetric spaces are called *locally weakly symmetric*.

Our goal is to prove the following two statements.

Theorem 1. *For $k \geq 2$, each locally k -fold symmetric Riemannian or pseudoriemannian manifold is locally symmetric.*

Theorem 2. *For $k \geq 2$, each locally k -fold symmetric Finslerian manifold is locally symmetric and Berwaldian.*

Since each locally $k+1$ -fold symmetric spaces is clearly also a locally k -fold symmetric, it is sufficient to prove both statements for $k = 2$ only. For $k = 1$ the statements are wrong, see [1] for a counterexample. In the global setting (i.e., when the manifold is complete), both statements were proved in [1]; the

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proof is quite involving, is essentially global, and in particular extensively uses the quite nontrivial results on weakly symmetric spaces from [3, 4, 5, 6]. Our proof is much easier, shorter, and works locally.

Let us explicitly mention that in the definition of locally k -fold symmetric metrics we do not require that the radius $r = r(x, V^k)$ from the definition is locally bounded from below; in fact, if it is locally bounded from below then (in the Riemannian and Finsler cases) the algebraic methods of [1] and [3, 4, 5, 6] are still applicable. But a consequence of our results is that locally ($k \geq 2$)-fold symmetric spaces are locally isometric to globally symmetric spaces so that the radius $r = r(x, V^k)$ is locally bounded from below.

All object in our paper are assumed to be sufficiently smooth; C^3 -smoothness is enough. The k -fold symmetric Finsler metrics F are automatically reversible. Theorem 2 remains true in the case of not necessary strictly convex Finsler metrics: indeed, similar to the end of the proof of [2, Theorem 9.2] one can show that if the theorem is true for strictly convex Finsler metrics it is true for arbitrary Finsler metrics.

1 Proof of Theorem 1

We denote by ∇R the covariant derivative of the Riemann curvature tensor with lowered indexes, ∇R is a $(0, 5)$ -tensorfield

$$\nabla R(U, V, X, Y, Z) = R_{ijkl,m} U^i V^j X^k Y^\ell Z^m.$$

Our goal is to show that $\nabla R = 0$, since Riemannian and pseudoriemannian metrics such that $\nabla R = 0$ are locally symmetric.

By the definition of locally 2-fold symmetric spaces, for any $x \in M$ and any two vectors $U, V \in T_x M$ there exists an isometry ϕ of $B_r(x)$ such that $\phi(x) = x$ and

$$d\phi_x(U) = -U, \quad d\phi_x(V) = -V. \quad (1)$$

We use this isometry and apply the Cartan's trick: since ∇R is preserved by the isometries, (1) implies

$$\begin{aligned} \nabla R(U, V, U, V, U) &= \nabla R(d\phi_x(U), d\phi_x(V), d\phi_x(U), d\phi_x(V), d\phi_x(U)) \\ &\stackrel{(1)}{=} \nabla R(-U, -V, -U, -V, -U) = -\nabla R(U, V, U, V, U), \end{aligned}$$

which implies that for any two vectors U, V we have $\nabla R(U, V, U, V, U) = 0$.

Replacing U by $U + tX + sY$ and V by $V + rZ$ in the left-hand side of the last equation we obtain a polynomial function in t, s, r ; vanishing the coefficient of tsr implies in view of the algebraic symmetries of the covariant derivative of the Riemann curvature tensor the equation

$$\begin{aligned} \nabla R(X, V, Y, Z, U) + \nabla R(Y, V, U, Z, X) + \nabla R(U, V, X, Z, Y) + \\ \nabla R(X, Z, Y, V, U) + \nabla R(Y, Z, U, V, X) + \nabla R(U, Z, X, V, Y) = 0. \end{aligned} \quad (2)$$

Since vanishing of ∇R implies that the metric is locally symmetric, Theorem 1 follows from the following linear algebraic lemma:

Lemma. *Suppose a $(0, 5)$ -tensor $\nabla R = R_{ijkl,m}$ has the algebraic symmetries of the covariant derivative of the Riemann curvature tensor (i.e., is skewsymmetric with respect to first two indexes, symmetric with respect to first and*

second pairs of indexes, satisfies the Bianchi equality with respect to the first three and with respect to the last three indexes). If in addition (2) is fulfilled, then it vanishes.

Proof. The equation (2) is a system of linear equations on the components of ∇R and its solution space is the kernel of the linear mapping

$$\nabla R \mapsto T, \quad T_{ijklm} := R_{ijkl,m} + R_{kijm,l} + R_{mijl,k} + R_{iljk,m} + R_{klmj,i} + R_{mlji,k}. \quad (3)$$

We show that this mapping has trivial kernel, or equivalently, that the mapping (3) has a left-inverse mapping. By direct calculation, we see that the mapping

$$T \mapsto \nabla R, \quad R_{jimk,\ell} = -\frac{1}{6}T_{ijlkm} - \frac{1}{12}T_{jkilm} + \frac{1}{12}T_{jlimk} + \frac{1}{6}T_{ijkml}. \quad (4)$$

is left inverse to (3), that is the superposition

$$\nabla R \xrightarrow{(3)} T \xrightarrow{(4)} \nabla R$$

is the identity. Lemma and therefore Theorem 1 are proved.

2 Proof of Theorem 2

We assume that our Finsler metric F is locally 2-fold symmetric; our first goal will be to show that it is *Berwaldian*, that is, there exists a torsion-free affine connection $\nabla = (\Gamma_{jk}^i)$ (called *the associated connection*) such that the parallel transport in this connection preserves the Finsler metric.

The proof is based on the same ideas as the proof of [2, Theorem 9.2], and will use the Binet-Legendre metric g_F introduced in [2, Definition 1.1], we recommend to have this paper by hand. In fact we are going to show that the Levi-Civita connection of g_F is the associated connection of F . It is sufficient to show that the spray coefficients $G^i(x, V)$ of F and the Christoffel symbols of g_F are related by

$$G^i(V) = \frac{1}{2}\Gamma_{jk}^i V^j V^k.$$

In order to do it, we consider an arbitrary point $x_0 \in M$ and work in the g_F -exponential coordinate system around x_0 ; in this coordinate system straight lines passing through x_0 are g_F -geodesics and $\Gamma(x_0)_{jk}^i \equiv 0$; our goal is to show that for any $V \in T_{x_0}M$ we have $G^i(V) = 0$. Consider the Finslerian geodesic $\gamma(t)$ such that $\gamma(0) = x_0$ and $\dot{\gamma}(0) = V$. It is known that every F -geodesic satisfies the equation $\ddot{\gamma}^i(t) + 2G^i(\gamma(t), \dot{\gamma}(t)) = 0$, therefore at x_0 we have

$$\ddot{\gamma}^i(0) + 2G^i(x_0, V) = 0.$$

Would $G^i(x_0, V) \neq 0$, at least one of the second derivatives of $\gamma^i(0)$ will be not zero and therefore the geodesic γ would have at most two intersections with any straight line passing through x_0 . Let us show that it has at least three intersections with some straight lines.

In order to do it, for $U \in T_{x_0}M$ we denote by ϕ^U a local isometry from the definition of locally k -fold symmetric spaces such that $\phi^U(x_0) = x_0$, $d\phi_{x_0}^U(\dot{\gamma}(0)) = -\dot{\gamma}(0)$ and $d\phi_{x_0}^U(U) = -U$. Since the sphere is compact,

there exists a finite number $\phi^{U_1}, \dots, \phi^{U_N}$ of such isometries such that for any $Z \in T_{x_0}M$ there exists $i \in \{1, \dots, N\}$ such that $d\phi_{x_0}^{U_i}(Z) \neq Z$. We can think that all these isometries operate on the ball of radius $r > 0$ around x_0 .

Take very small $\varepsilon > 0$ which in particular is much less than r and consider the points $\gamma(-\varepsilon)$ and $\gamma(\varepsilon)$. Denote by $\tilde{\gamma}$ the g_F -geodesic segment connecting $\gamma(-\varepsilon)$ and $\gamma(\varepsilon)$, and denote by C its g_F -midpoint, see Fig. 1.

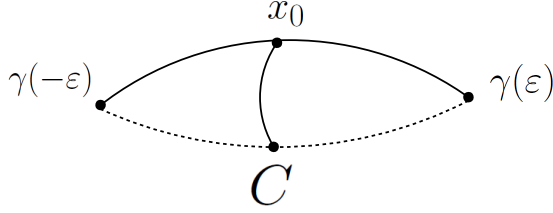


Fig.1. If the point C is different from x_0 , then the geodesic connecting the point x_0 with the point C is preserved by the isometry ϕ^{U_i} which contradicts that its initial velocity vector is not preserved by $d\phi_{x_0}^{U_i}$.

Let us show that $C = x_0$. Assume it is not the case. Denote by $Z \in T_{x_0}M$ the initial velocity vector of the F -geodesic connecting x_0 and C . Now, take the isometry ϕ^{U_i} such that $d\phi_{x_0}^{U_i}(Z) \neq Z$. The isometry is well-defined in the r -ball around x_0 , since all isometries $\phi^{U_1}, \dots, \phi^{U_N}$ are well-defined there. It sends the geodesic segment γ to itself with reversing the time, since $d\phi^{U_i}(\dot{\gamma}(0)) = -\dot{\gamma}(0)$. It therefore sends $\gamma(\varepsilon)$ to $\gamma(-\varepsilon)$ and $\gamma(-\varepsilon)$ to $\gamma(\varepsilon)$. Then, it sends the segment $\tilde{\gamma}$ to itself, which implies it sends its midpoint, which is C , to itself. Then, it sends the geodesic connecting x_0 and C to itself, which implies that its differential $d\phi_{x_0}^{U_i}$ sends Z to itself, which contradicts the choice of ϕ^{U_i} .

Thus, the geodesic $\tilde{\gamma}$ passes through x_0 which implies that it is a straight line in our coordinate system. By construction it has three intersections with γ , namely $\gamma(-\varepsilon)$, x_0 and $\gamma(\varepsilon)$, as we want. Finally, $G^i(x_0, V) = 0$.

Thus, the metric F is Berwaldian and the associated connection is the Levi-Civita connection of g_F . Since the associated connection is the Levi-Civita connection of a locally symmetric Riemannian manifold, the parallel transport in this connection commutes with the geodesic reflection. Then, the geodesic reflection is an isometry for F which implies that F is locally symmetric. Theorem 2 is proved.

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